Goal: Show Tate's conjecture for divisors + Artin's conjecture are "equivalent."

Artin's Conj: Let X be a proper scheme over $\mathbb{Z}$. Then $Br(X)$ is finite.

The Brauer Group of a local Ring

Let R be a local ring. Recall if k-field, $Br(k)$ is the collection of Morita equivalence classes of central simple algebras over k. The classification of these depends on the fact that $A \otimes A^{op} \xrightarrow{\sim} End_k(A)$ (A is simple! Count dimensions).

Def: Let (R, m) be a local ring, A an R-algebra. Then A is an Azumaya alg. over R if:
- A is a free R-module of finite rank,
- $A \otimes_R A^{op} \xrightarrow[\varphi]{\sim} End_R(A)$.

Prop: Let A be an Azumaya algebra over R. Then:
- $Z(A) = R$
- There is a bijection $\left\{\begin{matrix} \text{2-sided ideals} \\ \text{in } A \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \text{ideals in} \\ R \end{matrix}\right\}$.

Proof: Let $c \in Z(A)$. Then $\varphi(c \otimes 1 - 1 \otimes c) = 0$. Let $a_1, \dots, a_n$ be an R-basis of A, and hence $\{a_i \otimes a_j\}$ be an R-basis of $A \otimes_R A^{op}$. Hence $c \otimes 1 = 1 \otimes c \Rightarrow \Sigma r_i a_i \otimes 1 = 1 \otimes \Sigma r_i a_i$, which implies $c \in R$.

Now if $I \subset A$ is a two-sided ideal, any R-mod hom. $\psi: A \to A$, since $\psi \in End_R(A) = A \otimes_R A^{op}$, $\psi(I) \subset I$ as ψ is multiplication by some element. Using the hom's $x_i(\Sigma r_j a_j) = r_i \cdot 1$, we see $(I \cap R) \cdot A \subset I$.

Now let $J \subset R$, $\Sigma r_i a_i = x$, $r_i \in J$. Hence $x \in J \cdot A$. (insert non obvious manipulations) $(JA \cap R) \subset J$. ∎

Prop: If A is Azumaya$/R$ + R' is a commutative local R-alg. then $A \otimes_R R'$ is Azumaya$/R'$. If A is a free R-algebra of finite rank + $A/mA = \bar{A}_{R/m}$ is a CSA$/k = R/m$, then A is Azumaya$/R$.

Proof: The only non-obvious condition is the isomorphism.

$$\begin{tikzcd}
(A \otimes A^{op}) \otimes R' \arrow[r, "\sim"] \arrow[d, "s"] & End(A) \otimes R' \arrow[d, "s"] \\
(A \otimes R') \otimes (A^{op} \otimes R') \arrow[r, "\sim"] & End(A \otimes R')
\end{tikzcd}$$

want

Commutes ✓

The other statement follows from Nakayama. ∎

Corollary:
1) If A, A' are Azumaya$/R$, then $A \otimes_R A'$ is also.
2) $M_n(R)$ is Azumaya$/R$.

Def: Define $Br(R)$ to be Morita equivalence classes of Azumaya algebras over R (that is $A \otimes_R M_n(R) \simeq A' \otimes M_{n'}(R)$ for some n, n').

Morita Theory for CSA/k

We know CSA's A are isomorphic to $M_n(D)$ for some n, division algebra D.

Thm: (Morita) $\{A\text{-mod}\} \longleftrightarrow \{D\text{-mod}\}$.

Corollary: If A is a CSA/k, M + N are modules of the same $\dim/k$, then $M \cong N$ as A-mod.

Prop: Let A be Azumaya/R. Then every automorphism of A as an R-algebra is inner.
Proof: EC 1.4, Chap 4. ∎

Corollary: $\mathrm{Aut}_{R\text{-alg}}(M_n(R)) = PGL_n(R)$

Proof: We know $M_n(R)$ is Azumaya, hence all automorphisms are inner. Thus we have a surjection: $GL_n(R) \longrightarrow \mathrm{Aut}_{R\text{-alg}}(M_n(R))$, $u \mapsto (P \mapsto uPu^{-1})$, and its easy to see the kernel is R^*. ∎

Prop: If R is henselian, then the canonical map $Br(R) \longrightarrow Br(k)$, $A \mapsto \bar{A}$ is injective (in fact, an isomorphism.

Proof: 1.6, Chap 4 in EC. ∎

Corollary: If R is strictly Henselian, then $Br(R) = 0$.

Corollary: If R is Henselian, A is Az/R, then $\exists$ a finite étale $R \to R'$ of local algebras s.t. $A \otimes_R R' \cong M_n(R')$.

Schemes

Let X be a noetherian scheme. An $\mathcal{O}_X$-algebra A is assumed to be locally free of finite rank as an $\mathcal{O}_X$-module (not always true obviously). Assume also A_x is Azumaya over $\mathcal{O}_{X,x}$ for all closed $x \in X$. Note this assumption implies A_p is Azumaya for all $p \in X$.

Prop: Let A be an $\mathcal{O}_X$-algebra, which is coherent. Then TFAE:
1) A is Az/X,
2) A is locally free of finite rank over $\mathcal{O}_X$, and $A_x \otimes_{\mathcal{O}_{X,x}} k(x)$ is $csa/k(x)$ for all $x \in X$.
3) A is locally free as an $\mathcal{O}_X$-module, and $A \otimes_{\mathcal{O}_X} A^{op} \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_X\text{-mod}}\, A$.
4) $\exists$ a covering $(U_i \to X)$ in Et/X, s.t $\forall i$, $\exists r_i$, + an iso: $A \otimes_{\mathcal{O}_X} \mathcal{O}_{U_i} \cong M_{r_i}(\mathcal{O}_{U_i})$.
5) Same as (4) in the flat topology.

So A is a "family" of csa's over X, and is locally a matrix algebra. If $X = \mathrm{Spec}\, R$, then an Azumaya algebra over $X \longleftrightarrow$ f.g. projective R-module A with $A \otimes_R A^{op} \xrightarrow{\sim} \mathrm{End}_R(A)$.

Def: We say A + A' are equivalent if there are locally free $\mathcal{O}_X$-modules of finite rank $\mathcal{E}, \mathcal{E}'$ s.t. $A \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}(\mathcal{E}) \cong A' \otimes_{\mathcal{O}_X} \underline{\mathrm{End}}(\mathcal{E}')$. Equivalence classes form an associative + commutative monoid via $[A]\cdot[A'] = [A \otimes A']$, and defines a group $[A \otimes A^{op}] = [\underline{\mathrm{End}}(A)]$, the Brauer group of X, $Br(X)$.

We aim to give a cohomological description of this. Recall $Br(k) = Br(\mathrm{Spec}\, k) \simeq H^2(\mathrm{Gal}_k, k_s^*)$, but this is just $H^2_{et}(\mathrm{Spec}\, k, \mathbb{G}_m)$. So we hope $Br(X) = H^2_{et}(X, \mathbb{G}_m)$. Note $Br(-)$ is a contra. functor.

Prop: Let A be Az/X. Let ϕ be an automorphism of A as an $\mathcal{O}_X$-algebra. Then ϕ is inner, locally in the Zariski topology. That is, there is a Zariski-cover $\{U_i\}$ s.t. $\phi|_{U_i} : A|_{U_i} \circlearrowleft$ w/ $a \mapsto uau^{-1}$, $u \in \Gamma(U_i, A)^*$.

Proof: $\phi_x : A_x \xrightarrow{a \mapsto u_x a u_x^{-1}} A_x$ is inner by Skolem-Noether. Let $x \in U = \operatorname{Spec} R$ open. Then $u_x \in A_x = A(U) \otimes_R R_x$, can be written $\sum a_i r_i^{-1}$, $r_i \in R$. Replace R by $R[\{r_i^{-1}\}]$, so $\exists\ u \in A(\operatorname{Spec} R[\{r_i^{-1}\}])$ which lifts u_x, and is still a unit. Now on $V \subset U$, we want to compare $\phi(a) - uau^{-1}$, and since this holds at x, shrink until it holds on the open set again. ∎

Def: A sheaf PGL_n on X_{et} is defined as $PGL_n(U) = \operatorname{Aut}_{\Gamma(U, \mathcal{O}_U)}(M_n(\Gamma(U, \mathcal{O}_U)))$.

Hence we get $1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$ Skolem-Noether for sheaves.

$g \mapsto (M \mapsto gMg^{-1})$.

Fact: PGL_n is representable by a group scheme.

Assume X is quasi-proj. over $\operatorname{Spec} R$.

Cor: Have a short exact seq. of pointed sets $1 \to \check{H}^0(X_{et}, \mathbb{G}_m) \to \check{H}^0(GL_n) \to \check{H}^0(PGL_n) \to \check{H}^1(\mathbb{G}_m) \to \check{H}^1(GL_n) \to \check{H}^1(PGL_n) \to \check{H}^2(\mathbb{G}_m)$. (can't go further).

Thm: There is a natural (functorial) injection $Br(X) \hookrightarrow H^2(X_{et}, \mathbb{G}_m)$.
Proof: 2.5 in EC.

Cor: X regular, integral, q. compact. Then $Br(X) \subset Br(K(X))$.

Prop: The image of $H^1(X, PGL_n) \longrightarrow H^2(X, \mathbb{G}_m)$ is contained in the n-torsion subgroup. As a corollary, $Br(X)$ is a torsion group.

Proof:

$$\left.\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & \mu_n & \to & \mathbb{G}_m & \xrightarrow{x^n} & \mathbb{G}_m & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & SL_n & \to & GL_n & \xrightarrow{\det} & \mathbb{G}_m & \to & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & PGL_n & = & PGL_n & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}\right\} \Rightarrow \begin{array}{ccc}
H^1(PGL_n) & \longrightarrow & H^1(\mathbb{G}_m) \\
\| & & \uparrow \\
H^1(PGL_n) & \longrightarrow & H^2(\mu_n)
\end{array}$$

commutes

Torsion.

∎

There is a normal singular surface $/\mathbb{C}$ s.t. $H^2(X_{et}, \mathbb{G}_m)$ is not torsion. So in general $Br(X) \neq H^2(X, \mathbb{G}_m)$.

Q: • Is $Br(X) \longrightarrow H^2(X, \mathbb{G}_m)_{tor}$ surj for q. compact X?
• Is $Br(X) \longrightarrow H^2(X, \mathbb{G}_m)$ surj for regular X?

} Still open?

Conjecture (Artin): Let X be proper $/ \operatorname{Spec} \mathbb{Z}$. Then $|Br(X)| < \infty$.

For example, $X = \operatorname{Spec} \mathbb{Z}$. Then $Br(X) = 0$. If X = sm. proj. curve $/\mathbb{F}_q$, then $Br(X) = 0$. Still open for surfaces.

The Topological Case

Let $(X, \mathcal{O}_X)$ be a ringed space, with $\mathcal{O}_X$ the sheaf of cont. $\mathbb{C}$-valued functions on X. We can define an Azumaya alg. by a locally free $\mathcal{O}_X$-mod. of fin. rank s.t. $\forall p \in X$, the vector fiber is a matrix algebra. The same construction of the Brauer group goes through.

By considering the exponential sequence:

Prop: $Br(X) \subset H^3(X, \mathbb{Z})_{tor}$.

Thm (Serre): If X is a finite CW-complex, then $\sigma: Br(X) \to H^3(X, \mathbb{Z})_{tor}$ is an iso.

Tate's Conjecture

Take an abelian group M, ℓ a prime number. Define

$$M_\ell = \operatorname{Ker}(M \xrightarrow{\ell} M),$$
$$M_{\ell^n} = \operatorname{Ker}(M \xrightarrow{\ell^n} M),$$

and $M(\ell) = \bigcup M_{\ell^n}$, the ℓ-torsion part of M. Consider:

$$\begin{tikzcd}
M \arrow[r, "\ell"] \arrow[dr, bend left, "\ell^2"] \arrow[ddr, bend right, "\ell^3"'] & M \arrow[r] \arrow[d, "\ell"] & M/\ell \arrow[r] \arrow[d, "\ell"] & 0 \\
& M \arrow[r] \arrow[d, "\ell"] & M/\ell^2 \arrow[r] \arrow[d, "\ell"] & 0 \\
& M \arrow[r] & M/\ell^3 \arrow[r] & 0
\end{tikzcd}$$

Claim: $M/\ell^n = M \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^n$, $\varinjlim M/\ell^n = M \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_\ell$.

Def: The Tate module of M, $T_\ell(M)$ is the inverse limit of $\cdots \to M_{\ell^3} \xrightarrow{\ell} M_{\ell^2} \xrightarrow{\ell} M_\ell \to 0$.

Def: An ℓ-torsion abelian group $M = M(\ell)$ is of cofinite rank if M_ℓ is finite. Equivalently: $M \cong (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^r \times$ fin. ℓ-group.

Note this implies $T_\ell(M)$ is a fin. free module of rank r, and $T_\ell(M) = 0$ if M is finite.

Picard Variety

$\operatorname{Pic} X$ is a contravariant functor in X, is it representable by a scheme? Over $\mathbb{C}$, we have the exponential sequence (for smooth X): $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 0$, and in cohomology, applying the Hodge decomposition: $H^1(X, \mathbb{Z})/\text{torsion}$ is a maximal lattice in $H^1(X, \mathbb{C})$. Hence the quotient is a torus, called a complex Abelian variety. This group is denoted $\operatorname{Pic}^\circ(X)$.

If X is sm. proj. $/k = \bar{k}$, we get the same sequence $0 \to \operatorname{Pic}^\circ(X) \to \operatorname{Pic}(X) \to \underset{\text{f.g.}}{NS(X)} \to 0$, and $\operatorname{Pic}^\circ(X)$ is again an abelian variety, and is ℓ-divisible if $\ell \neq \operatorname{char} k$. If $k = \mathbb{F}_q$, then $\operatorname{Pic}(X)$ is f.g.

Note the slogan: "$Br(X)$ parametrizes nonalgebraic cohomology classes in $H^2(X)$". Indeed the Kummer sequence gives $0 \to \operatorname{Pic}(X)/\ell^n \to H^2(\mu_{\ell^n}) \to H^2(\mathbb{G}_m)_{\ell^n} \to 0$, and taking the inverse limit:

$$\begin{array}{ccccccc}
0 \to & \varprojlim \operatorname{Pic}(X)/\ell^n & \to & \varprojlim H^2(\mu_{\ell^n}) & \to & T_\ell(H^2(\mathbb{G}_m)) & \to 0 \\
& \parallel & & \parallel & & \parallel & \\
& \operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell & & H^2(X, T_\ell(\mu)) & & T_\ell(Br(X)) &
\end{array}$$

Some work w/ $\bar{X}$ gives:

<u>Conj:</u> $cl_{\bar{X}} : \mathrm{Pic}(X) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} H^2(\bar{X}, \mathbb{Q}_\ell(1))^{\mathrm{Gal}(k)}$.

<u>Thm (Artin, Tate):</u> Let X be sm. proj. $/\mathbb{F}_q$. Then Tate's conjecture for $X \iff \bigoplus_{\ell \neq p} \mathrm{Br}(X)(\ell)$, $q = p^n$, is finite.